

# Algebraic treatment of $\mathcal{PT}$ -symmetric coupled oscillators

Francisco M. Fernández

The purpose of this paper is the discussion of a pair of coupled linear oscillators that has recently been proposed as a model of a system of two optical resonators. By means of an algebraic approach we show that the frequencies of the classical and quantum-mechanical interpretations of the optical phenomenon are exactly the same. Consequently, if the classical frequencies are real, then the quantum-mechanical eigenvalues are also real.

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## I. INTRODUCTION

In a recent paper Bender et al[1] discussed the classical and quantum-mechanical versions of a system of two coupled linear oscillators one with gain and the other with loss. When the gain and loss parameters are equal the Hamiltonian derived from the classical equations of motion is  $\mathcal{PT}$ -symmetric and exhibits two  $\mathcal{PT}$ -transitions in terms of the coupling parameter. In the unbroken- $\mathcal{PT}$  region the classical frequencies are real. This analysis is straightforward because one can derive analytical expressions for such frequencies from the equations of motion. On the other hand, the analysis of the quantum-mechanical model does not appear to be so simple. Although the authors obtained analytical expressions for the eigenvalues and eigenfunctions, they estimated the regions of broken- and unbroken- $\mathcal{PT}$  symmetry numerically and conjectured that both the eigenvalues of the quantum-mechanical Hamiltonian and the classical frequencies are real in the same region of model parameters. This theoretical investigation was motivated by recent experiments on a  $\mathcal{PT}$ -symmetric system of two coupled optical resonators[2].

The purpose of the present paper is to derive an analytic clear connection between the classical frequencies on the one hand and the quantum-mechanical energies on the other. In section II we apply a well known algebraic method and show that the quantum-mechanical frequencies (spacing between eigenvalues) are exactly the classical ones. We also write the quantum-mechanical energies in terms of the classical frequencies and analyse the spectrum. Finally, in section III we review the main results of the paper and draw conclusions.

## II. QUANTUM-MECHANICAL MODEL

As indicated above, the analysis of the classical model is straightforward and here we focus on the quantal version in the case of equal gain and loss. The Hamiltonian operator derived from the classical equations of motion is given by[1]

$$H = p_x p_y + \gamma(y p_y - x p_x) + (\omega^2 - \gamma^2) xy + \frac{\epsilon}{2}(x^2 + y^2), \quad (1)$$

where  $p_x$  and  $p_y$  are the momenta conjugate to coordinates  $x$  and  $y$ :  $[x, p_x] = [y, p_y] = i$ . It is  $\mathcal{PT}$  symmetric  $\mathcal{P}TH\mathcal{PT} = H$  since it is invariant under the combined effect of parity  $\mathcal{P} : \{x, y, p_x, p_y\} \rightarrow \{-y, -x, -p_y, -p_x\}$  and time reversal  $\mathcal{T} : \{x, y, p_x, p_y\} \rightarrow \{x, y, -p_x, -p_y\}$ [1]. Note that we can also choose  $\mathcal{P} : \{x, y, p_x, p_y\} \rightarrow \{y, x, p_y, p_x\}$  for exactly the same purpose. In the language of point-group symmetry such *parity* transformations are given by

reflection planes  $\sigma_v$  and  $\sigma'_v$  perpendicular to the  $x - y$  plane[3, 4]. Since the Hamiltonian  $H$  is also invariant under inversion  $\hat{i} : \{x, y, p_x, p_y\} = \{-x, -y, -p_x, -p_y\}$  its eigenvectors  $|\psi\rangle$  satisfy  $\hat{i}|\psi\rangle = \pm|\psi\rangle$ . This result is consistent with the solutions of the form  $P_{mn}(x, y)e^{-(2axy+bx^2+cy^2)}$  obtained by Bender et al[1] in the coordinate representation, where the polynomials  $P_{mn}(x, y)$  satisfy  $P_{mn}(-x, -y) = (-1)^m P_{mn}(x, y)$ . Point-group symmetry proved to be useful for the discussion of broken- and unbroken- $\mathcal{PT}$  symmetry in some anharmonic oscillators[5, 6]. As far as we know Klaiman and Cederbaum[5] were the first to construct a non-Hermitian Hamiltonian with consecutive  $\mathcal{PT}$  phase transitions in terms of the coupling parameter.

It is worth noting that the Hamiltonian (1) is also a symmetric operator  $\langle H\psi | \varphi \rangle = \langle \psi | H\varphi \rangle$ , which for brevity we formally express by means of the usual notation for Hermitian operators as  $H^\dagger = H$ . In particular, note that  $(yp_y - xp_x)^\dagger = (p_y y - p_x x) = (yp_y - xp_x)$ . Therefore, one expects real eigenvalues at least for some range of the model parameters  $\omega$ ,  $\gamma$  and  $\epsilon$ .

In order to solve the eigenvalue equation  $H|\psi\rangle = E|\psi\rangle$  in a way that clearly reveals the connection with the classical interpretation we resort to a well known algebraic method[7]. It is suitable when there exists a set of symmetric operators  $\{O_1, O_2, \dots, O_N\}$  that satisfy the commutation relations

$$[H, O_i] = \sum_{j=1}^N H_{ji} O_j. \quad (2)$$

We look for an operator of the form

$$Z = \sum_{i=1}^N c_i O_i, \quad (3)$$

such that

$$[H, Z] = \lambda Z. \quad (4)$$

The operator  $Z$  is important for our purposes because

$$HZ|\psi\rangle = (E + \lambda)Z|\psi\rangle. \quad (5)$$

It follows from equations (2), (3) and (4) that

$$(\mathbf{H} - \lambda \mathbf{I})\mathbf{C} = 0, \quad (6)$$

where  $\mathbf{H}$  is an  $N \times N$  matrix with elements  $H_{ij}$ ,  $\mathbf{I}$  is the  $N \times N$  identity matrix, and  $\mathbf{C}$  is an  $N \times 1$  column matrix with elements  $c_i$ .  $\mathbf{H}$  is called the adjoint or regular matrix representation of  $H$  in the operator basis  $\{O_1, O_2, \dots, O_N\}$ [7]. In the case of an Hermitian operator we expect all the roots  $\lambda_i$ ,  $i = 1, 2, \dots, N$  to be real. These roots are obviously the natural frequencies of the quantum-mechanical system. Here we apply the same approach to symmetric Hamiltonians because all the relevant equations are formally identical. If  $\lambda$  is real then it follows from equation (4) that

$$[H, Z^\dagger] = -\lambda Z^\dagger, \quad (7)$$

where  $Z^\dagger$  is a linear combination like (3) with coefficients  $c_i^*$ . This equation tells us that if  $\lambda$  is a real root of  $\det(\mathbf{H} - \lambda \mathbf{I}) = 0$ , then  $-\lambda$  is also a root. Obviously,  $Z$  and  $Z^\dagger$  are a pair of annihilation-creation or ladder operators because, in addition to (5), we also have

$$HZ^\dagger|\psi\rangle = (E - \lambda)Z^\dagger|\psi\rangle. \quad (8)$$

Other authors have already applied Lie-algebraic methods to the Hamiltonian (1) without coupling ( $\epsilon = 0$ ) [8–10] but they were not interested in the quantum-mechanical frequencies.

In the present case, the obvious choice  $\{O_1, O_2, O_3, O_4\} = \{x, y, p_x, p_y\}$  leads to the matrix representation

$$\mathbf{H} = i \begin{pmatrix} \gamma & 0 & \epsilon & \omega^2 - \gamma^2 \\ 0 & -\gamma & \omega^2 - \gamma^2 & \epsilon \\ 0 & -1 & -\gamma & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}, \quad (9)$$

with characteristic polynomial

$$\lambda^4 + \lambda^2 (4\gamma^2 - 2\omega^2) - \epsilon^2 + \omega^4 = 0, \quad (10)$$

that is exactly the one that yields the classical frequencies [1]. Two of its roots are

$$\begin{aligned} \lambda_1 &= \sqrt{\sqrt{\epsilon^2 + 4\gamma^4 - 4\gamma^2\omega^2} - 2\gamma^2 + \omega^2} \\ \lambda_2 &= \sqrt{-\sqrt{\epsilon^2 + 4\gamma^4 - 4\gamma^2\omega^2} - 2\gamma^2 + \omega^2}, \end{aligned} \quad (11)$$

and the other two ones are  $\lambda_3 = -\lambda_1$  and  $\lambda_4 = -\lambda_2$  in agreement with the more general equations (4) and (7). The operators  $Z_1$  and  $Z_2$  associated to  $\lambda_1$  and  $\lambda_2$  are creation or rising, while  $Z_2 = Z_1^\dagger$  and  $Z_3 = Z_2^\dagger$  are annihilation or lowering. The classical and quantal frequencies are exactly the same because the relevant Poisson brackets and commutators are similar:  $i\{H, O_i\} \rightarrow [H, O_i]$ . This result reveals why the condition for real classical frequencies

$$2\gamma\sqrt{\omega^2 - \gamma^2} < \epsilon < \omega^2 \quad (12)$$

is also the condition for real spectrum (unbroken- $\mathcal{PT}$  region) in the quantum-mechanical counterpart [1]. If we write the polynomial equation (10) as  $(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2) = 0$  then we realize that

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 &= 2\omega^2 - 4\gamma^2 \\ \lambda_1^2 \lambda_2^2 &= \omega^4 - \epsilon^2. \end{aligned} \quad (13)$$

Throughout this paper we keep the parameter  $\omega$  in order to facilitate the discussion of the results of Bender et al. However, it is worth noting that we can choose  $\omega = 1$  without loss of generality as follows from the transformation  $\{\lambda, a, \omega, \gamma, \epsilon\} \rightarrow \{\frac{\lambda}{\omega}, \frac{a}{\omega}, 1, \frac{\gamma}{\omega}, \frac{\epsilon}{\omega^2}\}$ .

Bender et al [1] derived the energies

$$E_{mn} = (m+1)a + (2n-m)\Delta, \quad (14)$$

where  $m = 0, 1, \dots$  and  $n = 0, 1, \dots, m$ . In this equation  $a$  is a root of

$$4a^4 + 4a^2 (2\gamma^2 - \omega^2) + \epsilon^2 + 4\gamma^2 (\gamma^2 - \omega^2) = 0, \quad (15)$$

and

$$\begin{aligned} \Delta &= \sqrt{bc - \gamma^2}, \\ b &= c^* = \frac{\epsilon}{2(a + i\gamma)}. \end{aligned} \quad (16)$$

The expressions for  $a$ ,  $b$  and  $c$  come from solving the eigenvalue equation  $H|\psi_{00}\rangle = E_{00}|\psi_{00}\rangle$  in the coordinate representation with the ansatz  $\psi_{00}(x, y) = e^{-(bx^2 + cy^2 + 2axy)}$ , procedure that also yields  $E_{00} = a$ . If we write  $\xi = a^2$  then we obtain the roots

$$\begin{aligned}\xi_1 &= \frac{\omega^2 - 2\gamma^2 - \sqrt{\omega^4 - \epsilon^2}}{2}, \\ \xi_2 &= \frac{\omega^2 - 2\gamma^2 + \sqrt{\omega^4 - \epsilon^2}}{2}.\end{aligned}\tag{17}$$

Following Bender et al we write  $a_1 = -\sqrt{\xi_1}$ ,  $a_2 = \sqrt{\xi_1}$ ,  $a_3 = -\sqrt{\xi_2}$ ,  $a_4 = \sqrt{\xi_2}$ . For concreteness, from now on we choose

$$a = a_2 = \frac{1}{2}\sqrt{2\omega^2 - 4\gamma^2 - 2\sqrt{\omega^4 - \epsilon^2}}.\tag{18}$$

So far, we have shown that the classical and quantum-mechanical interpretations exhibit exactly the same frequencies; it only remains to rewrite the eigenvalues (14) in terms of these frequencies. One can easily verify that

$$\lambda_1 = \Delta + a, \lambda_2 = \Delta - a,\tag{19}$$

is consistent with (13). Thus, the expression for the energies becomes

$$\begin{aligned}E_{m,n} &= \frac{\lambda_1(2n+1)}{2} - \frac{\lambda_2(2m-2n+1)}{2} = \\ &= n\lambda_1 + (n-m)\lambda_2 + a.\end{aligned}\tag{20}$$

Bender et al[1] showed numerically that  $a_2$  and  $\Delta$  are real and positive in the unbroken- $\mathcal{PT}$  region and concluded that the eigenvalues are also real and positive. However, this is not the case because  $a - \Delta < 0$  and for every value of  $n$   $E_{mn} \rightarrow -\infty$  as  $m \rightarrow \infty$ . As an example, consider the model parameters  $\omega = 1$ ,  $\gamma = 0.05$  and  $\epsilon = 0.5$  that lie the unbroken- $\mathcal{PT}$  region. In this case  $\Delta \approx 0.964630863$  and  $a \approx 0.2539434939$  that confirms what we have just said.

We can analyse those results by means of the algebraic method. If  $E_{00}$  were the lowest eigenvalue then both  $Z_3\psi_{00}$  and  $Z_4\psi_{00}$  would be expected to vanish. However, for the parameters chosen above we found that  $Z_2\psi_{00}$  and  $Z_3\psi_{00}$  vanish while  $Z_1\psi_{00}$  and  $Z_4\psi_{00}$  do not. Therefore, the eigenfunctions are given by

$$\psi_{nk} = Z_1^n Z_4^k \psi_{00}\tag{21}$$

with eigenvalues  $E_{nk} = n\lambda_1 - k\lambda_2 + a$  wich agree with (20) if  $k = m - n$ . The algebraic method clearly shows that the spectrum is unbounded from below.

### III. CONCLUSIONS

The main purpose of this paper is to show that part of the mathematical analysis of some classical systems also applies to their quantum-mechanical counterparts. The underlying connection is that the frequencies of both interpretations are exactly the same because of the similarity between the Poisson brackets and commutators. Therefore, if the frequencies of the motion of the classical system are real then the quantum-mechanical eigenvalues are also real. The quantum-mechanical frequencies are the eigenvalues of the regular or adjoint matrix representation of the

Hamiltonian operator in a suitable basis set of operators, whereas the corresponding eigenvectors provide the ladder operators. This well known algebraic approach is suitable for many problems[7] and in particular for Hamiltonians that are quadratic functions of the coordinates and their conjugate momenta. Such Hamiltonians are suitable models for many physical problems like the one that motivated the paper by Bender et al[1], among others[8–10].

In closing it is worth mentioning that the fact that the spectrum of the Hamiltonian (1) is not strictly positive, contrary to what Bender et al assumed, does not appear to be relevant to the interpretation of the physical data which is fitted by the classical (and also quantal) frequencies[2].

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